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Some Relations on Laguerre Matrix Polynomials

Ayman Shehata

Department of Mathematics, Faculty of Science, Assiut University, Assiut 71516, Egypt and Department of Mathematics, College of Science and Arts, Unaizah, Qassim University, Qassim, Kingdom of Saudi Arabia.

E-mail: drshehata2006@yahoo.com

ABSTRACT

The main object of this paper is to give a different approach to proof of generating matrix functions for Laguerre matrix polynomials. We also obtain the hypergeometric matrix representations, addition theorem, finite summation formula and an integral representation for Laguerre matrix polynomials. We get the relations between Laguerre, Legendre and Hermite matrix polynomials. We get the generating matrix functions for the Laguerre matrix polynomials, involving the Horn's and hypergeometric matrix functions. Finally, we define a new generalization of the Laguerre matrix polynomials with the hypergeometric matrix function.

Keywords: Matrix functions, Laguerre, Legendre and Hermite matrix polynomials, Hypergeometric matrix functions, Generating matrix function, Finite summation, Integral representation.

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1. Introduction

Orthogonal matrix polynomials are becoming more and more relevant in the two last decades. Classical orthogonal polynomials are extended to orthogonal matrix polynomials by Aktaş et al. (2012), Khammash and Shehata (2012), Metwally et al. (2008), Shehata (2011)-Shehata (2015), and some results in the theory of classical orthogonal polynomials are extended to orthogonal matrix polynomials, see Al-Gonah (2014), Altin and Çekim (2012), Çekim et al. (2011), Çekim et al. (2013), Defez (2013), Jódar and Defez (1998), Kargin and Kurt (2013), Metwally and Shehata (2013), Shehata (2009) and Yasmin (2014). Laguerre matrix polynomials have been introduced and studied in Jódar and Defez (1998), Jódar and Sastre (1998), Jódar and Sastre (2001), Jódar and Sastre (2004), Sastre and Defez (2006), Sastre Defez and Jódar (2006), Sastre and Jódar (2006). As in the corresponding problem for scalar functions, the problem of the development of matrix functions in a series of Laguerre matrix polynomials requires some new results about the matrix operational calculus not available in the literature. From this motivation, we prove some new properties for the Laguerre matrix polynomials. The outline of this paper is as follows: In Section 2, we give addition, summation formulas, and a different approach to proof of generating matrix functions of the Laguerre matrix polynomials and write these polynomials as hypergeometric matrix functions. Furthermore, we show the integral representation for Laguerre matrix polynomials. We get expansions of the Laguerre matrix polynomials as series of Hermite and Legendre matrix polynomials in Section 3. We get some results which follow from this new generating matrix function, involving the Horn's matrix functions of two variables and hypergeometric matrix functions of three variables in Section 4. Finally, we define the generalized Laguerre matrix polynomials of two variables with the hypergeometric matrix function.

Throughout this paper, if A is a matrix in $\mathbb{C}^{r\times r}$, its spectrum $\sigma(A)$ denotes the set of all the eigenvalues of A. If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z, which are defined in an open set Ω of the complex plane and A, B are matrices in $\mathbb{C}^{r\times r}$ such that $\sigma(A)\subset\Omega$, $\sigma(B)\subset\Omega$ then from the properties of matrix functional calculus (see Dunford and Schwartz (1957)), it follows that: $f(A)g(B)=g(B)f(A)$, where $AB=BA$.

If y is a complex number with $|y|<1$ and a is a complex number, then $g(a)=(1-y)^{-a}=\sum_{n=0}^{\infty}\frac{(a)_n}{n!}y^n$ is an holomorphic function in $\mathbb{C}^{r\times r}$. Therefore, applying the holomorphic functional calculus (see Dunford and Schwartz

(1957)) to any matrix A in $\mathbb{C}^{r\times r}$, the image of g, acting on A yields

$$g(A) = (1-y)^{-A} = \sum_{n=0}^{\infty} \frac{(A)_n}{n!} y^n; \quad |y| < 1, \tag{1}$$

where $(A)_n$ is the Pochhammer symbol or shifted factorial which is defined by

$$(A)_n = A(A+I)(A+2I)\dots(A+(n-1)I); n \geq 1, \tag{2}$$

with $(A)_0 = I$. It is easy to show that

$$(A)_{n+k} = (A)_n (A+nI)_k. \tag{3}$$

By using (2), it is easy to find that

$$(A)_{n-k} = (-1)^k (A)_n [(I-A-nI)_k]^{-1}; \quad 0 \leq k \leq n \tag{4}$$

and

$$(-nI)_k = \frac{(-1)^k \, n!}{(n-k)!} I, 0 \leq k \leq n. \tag{5}$$

So generalized form of equation (1) which is called hypergeometric matrix function ${}_2F_1(A, B; C; z)$, is defined by (see Jódar, and Cortes (1998))

$${}_2F_1\left(A, B; C; z\right) = \sum_{n \geq 0} \frac{(A)_n (B)_n [(C)_n]^{-1}}{n!} z^n; |z| < 1, \tag{6}$$

for matrices A, B, C in $\mathbb{C}^{r\times r}$ such that

$$C + nI \text{is an invertible matrix for all integers } n \geq 0. \tag{7}$$

Definition 1.1. *Let A be a matrix in $\mathbb{C}^{r\times r}$ such that*

$$-k \notin \sigma(A) \text{ for every integer } k > 0, \tag{8}$$

and λ is a complex number with $\Re(\lambda) > 0$, then the Laguerre matrix polynomials are defined by (see Jódar et al. (1994))

$$L_n^{(A,\lambda)}(x) = \sum_{k=0}^{n} \frac{(-1)^k (A+I)_n [(A+I)_k]^{-1} (\lambda x)^k}{k!(n-k)!} \tag{9}$$

and satisfied the Rodrigues formula

$$L_n^{(A,\lambda)}(x) = \frac{x^{-A} e^{\lambda x}}{n!} \frac{d^n}{dx^n} [x^{A+nI} e^{-\lambda x}], n \geq 0. \tag{10}$$

According to Jódar et al. (1994), Laguerre matrix polynomials are generated by

$$\sum_{n=0}^{\infty} L_n^{(A,\lambda)}(x)\, t^n = (1-t)^{-(A+I)} \exp\left(\frac{-\lambda x t}{1-t}\right), \tag{11}$$

where $t \in \mathbb{C}, |t| < 1, x \in \mathbb{C}$.

Furthermore, Çekim and Altin deal with multiplication formula; (see Çekim and Altin (2013))

$$L_n^{(A,\lambda)}(xy) = \sum_{k=0}^{n} \frac{(A+(k+1)I)_{n-k}}{(n-k)!} y^k (1-y)^{n-k} L_k^{(A,\lambda)}(x)$$

and a new generating matrix function including hypergeometric matrix function;

$$\sum_{n=0}^{\infty} (B)_n \left[(A+I)_n\right]^{-1} L_n^{(A,\lambda)}(x)\, t^n = (1-t)^{-B}\ {}_1F_1\left(B; A+I; \frac{-\lambda x t}{1-t}\right). \tag{12}$$

Definition 1.2. *Let A, B and C be matrices in $\mathbb{C}^{r\times r}$ such that $C+(m+n)I$ is an invertible matrix for all integers $m+n \geq 0$. Then the Horn's matrix functions H_6 and Φ_3 of two variables are defined by (see Shehata (2009))*

$$\begin{aligned} H_6(A;C;z,w) &= \sum_{m,n=0}^{\infty} \frac{(A)_{2m+n}[(C)_{m+n}]^{-1}}{m!n!} z^m w^n, \\ \Phi_3(B;C;z,w) &= \sum_{m,n=0}^{\infty} \frac{(B)_m[(C)_{m+n}]^{-1}}{m!n!} z^m w^n. \end{aligned} \tag{13}$$

Definition 1.3. *Let A_1, B_1, B_2, C_1 and C_2 be matrices in $\mathbb{C}^{r\times r}$ such that C_1+mI is an invertible matrix for all integers $m \geq 0$ and $C_2+(n+p)I$ is an invertible matrix for all integers $n+p \geq 0$. Then the hypergeometric matrix functions ${}_3\Phi_M^{(4)}$ and ${}_3\Phi_G^{(1)}$ of three variables are defined as follows (see Shehata (2009) and Shehata (2014 b))*

$$\begin{aligned} &{}_3\Phi_M^{(4)}(A_1,B_1,B_1;C_1,C_2,C_2;z,w,u) \\ &\quad = \sum_{m,n,p=0}^{\infty} \frac{(A_1)_m(B_1)_{m+p}[(C_1)_m]^{-1}[(C_2)_{n+p}]^{-1}}{m!n!p!} z^m w^n u^p, \\ &{}_3\Phi_G^{(1)}(A_1,A_1,A_1,B_1,B_2;C_1,C_2,C_2;z,w,u) \\ &\quad = \sum_{m,n,p=0}^{\infty} \frac{(A_1)_{m+n+p}(B_1)_m(B_2)_n[(C_1)_m]^{-1}[(C_2)_{n+p}]^{-1}}{m!n!p!} z^m w^n u^p. \end{aligned} \tag{14}$$

We conclude this section recalling a result related to the rearrangement of the terms in iterated series. If $A(k,n)$ and $B(k,n)$ are matrices in $\mathbb{C}^{r\times r}$ for $n \geq 0$ and $k \geq 0$, then in an analogous way to the proof of Lemma 11 (see Rainville (1962)), it follows that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{2}n]} A(k,n-2k) \tag{15}$$

and

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k,n-k). \tag{16}$$

Similarly, we can write

$$\begin{aligned} \sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k,n) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n+k), \\ \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{2}n]} A(k,n) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n+2k). \end{aligned} \tag{17}$$

2. Some relations on Laguerre matrix polynomials

In this section, we obtain some generating functions, new results and relations for Laguerre matrix polynomials and write these polynomials as hypergeometric matrix functions. Moreover we give integral representation of Laguerre matrix polynomials. We write Laguerre Matrix Polynomials as hypergeometric matrix functions in the following theorems.

Theorem 2.1. *Let A be a matrix in $\mathbb{C}^{r\times r}$ satisfying the spectral condition (8) and let λ be a complex number with $\Re(\lambda) > 0$. Then Laguerre matrix polynomials can be written as hypergeometric matrix function:*

$$L_n^{(A,\lambda)}(x) = \frac{(A+I)_n}{n!}\ {}_1F_1\Big(-nI; A+I; \lambda x\Big). \tag{18}$$

Proof. From (9) and using the relation (5), the equation (18) as follows directly. □

Theorem 2.2. *Let A be a matrix in $\mathbb{C}^{r\times r}$ satisfying the spectral condition (8) and let λ be a complex number with $\Re(\lambda) > 0$. For Laguerre matrix polynomials, we have hypergeometric matrix representation as follows:*

$$L_n^{(A,\lambda)}(x) = \frac{(-\lambda x)^n}{n!}\ {}_2F_0\bigg(-nI, -A - nI; -; -\frac{1}{\lambda x}\bigg). \tag{19}$$

Proof. Taking $n - k$ instead of k in (9), one gets

$$L_n^{(A,\lambda)}(x) = \sum_{k=0}^{n} \frac{(-1)^{n-k}\,(A+I)_n\,\left[(A+I)_{n-k}\right]^{-1}(\lambda x)^{n-k}}{k!(n-k)!}$$

and using the relations (4) and (5), we get (19). □

In the following theorem, we prove the addition formula of the Laguerre matrix polynomials.

Theorem 2.3. *Let A and B be matrices in $\mathbb{C}^{r\times r}$ satisfying the spectral condition (8) and let λ be a complex number with $\Re(\lambda) > 0$, then the Laguerre matrix polynomials satisfy the addition formula as follows:*

$$L_n^{(A+B+I,\lambda)}(x+y) = \sum_{k=0}^{n} L_k^{(A,\lambda)}(x)\,L_{n-k}^{(B,\lambda)}(y), \tag{20}$$

Proof. From equation (11) and the fact that

$$\begin{aligned}&(1-t)^{-(A+B+2I)}\exp\left(\frac{-\lambda(x+y)t}{1-t}\right)\\ &= (1-t)^{-(A+I)}\exp\left(\frac{-\lambda xt}{1-t}\right)(1-t)^{-(B+I)}\exp\left(\frac{-\lambda xt}{1-t}\right)\end{aligned}$$

we have

$$\begin{aligned}\sum_{n=0}^{\infty} L_n^{(A+B+I,\lambda)}(x+y)\,t^n &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} L_k^{(A,\lambda)}(x)\,L_n^{(B,\lambda)}(y)\,t^{n+k}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} L_k^{(A,\lambda)}(x)\,L_{n-k}^{(B,\lambda)}(y)\,t^n.\end{aligned}$$

Comparing the coefficients of t^n, we obtain (20). □

Remark 2.1. *(i) Putting $A = A - I$ and $B = A - I$ in equation (20), we get the following recently result [Çekim (2013), p.820]:*

$$L_n^{(2A-I,\lambda)}(x+y) = \sum_{k=0}^{n} L_k^{(A-I,\lambda)}(x)\, L_{n-k}^{(A-I,\lambda)}(y).$$

(ii) Putting $A = B - I$, $B = A - I$ and $x = y$ in equation (20), we get the following recently result [Çekim (2013), p.820]:

$$L_n^{(A+B-I,\lambda)}(2x) = \sum_{k=0}^{n} L_k^{(B-I,\lambda)}(x)\, L_{n-k}^{(A-I,\lambda)}(x).$$

To show the availability of equation (12) we give the following theorem.

Theorem 2.4. *Let A and B be commuting matrices in $\mathbb{C}^{r\times r}$ satisfying the spectral condition (8) and let λ be a complex number with $\Re(\lambda) > 0$, then the Laguerre matrix polynomials have the following relation:*

$$\begin{aligned} L_n^{(A,\lambda)}(x) &= (A+I)_n \left[(B)_n\right]^{-1} \\ &\times \sum_{k=0}^{n} (A+I-B)_k \left[(A+I)_k\right]^{-1} L_k^{(A,\lambda)}(-x)\, L_{n-k}^{(2B-A-2I,\lambda)}(x). \end{aligned} \tag{21}$$

Proof. One can easily show that

$${}_1F_1\left(B; A+I; \frac{-\lambda x t}{1-t}\right) = \exp\left(\frac{-\lambda x t}{1-t}\right)\ {}_1F_1\left(A+I-B; A+I; \frac{\lambda x t}{1-t}\right). \tag{22}$$

Using (22) in (12), we get

$$\begin{aligned} &\sum_{n=0}^{\infty} (B)_n \left[(A+I)_n\right]^{-1} L_n^{(A,\lambda)}(x)\, t^n \\ &= (1-t)^{-B} \exp\left(\frac{-\lambda x t}{1-t}\right)\ {}_1F_1\left(A+I-B; A+I; \frac{\lambda x t}{1-t}\right) \\ &= \left[\sum_{n=0}^{\infty} L_n^{(2B-A-2I,\lambda)}(x)\, t^n\right] \left[\sum_{k=0}^{\infty} (A+I-B)_k \left[(A+I)_k\right]^{-1} L_k^{(A,\lambda)}(-x)\, t^k\right]. \end{aligned}$$

With aid of (16), we have (21). □

Substituting $B = A + I$ in equation (21), we get the following corollary.

Corollary 2.1. *Let A be a matrix in $\mathbb{C}^{r\times r}$ satisfying the spectral condition (8) and let λ be a complex number with $\Re(\lambda) > 0$, then the Laguerre matrix polynomials have the following summation formula:*

$$L_n^{(A,\lambda)}(x) = \sum_{k=0}^{n} [(A+I)_k]^{-1} L_k^{(A,\lambda)}(-x) L_{n-k}^{(A,\lambda)}(x).$$

We shall perform this transformation to exhibit the technique. If we replace x by $\frac{x}{1-t}$, t by $-\frac{yt}{1-t}$ and multiply both sides of (12) by $(1-t)^{-(A+I)}\exp\left(\frac{-\lambda xt}{1-t}\right)$, we get

$$\begin{aligned}
&(1-t)^{-(A+I)}\left[1+\frac{yt}{1-t}\right]^{-B}\exp\left(\frac{-\lambda xt}{1-t}\right)\ {}_1F_1\left(B; A+I; \frac{\frac{\lambda xyt}{(1-t)^2}}{1+\frac{yt}{1-t}}\right)\\
&= \sum_{n=0}^{\infty}(B)_k[(A+I)_k]^{-1}(1-t)^{-(A+I+B)}\exp\left(\frac{-\lambda xt}{1-t}\right) L_k^{(A,\lambda)}(\frac{x}{1-t})(-1)^k y^k t^k\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-n)_k(B)_k}{k!}[(A+I)_k]^{-1}L_n^{(A,\lambda)}(x)y^k t^n\\
&= \sum_{n=0}^{\infty}\ {}_2F_1\Big(-nI, B; A+I; y\Big)L_n^{(A,\lambda)}(x)t^n.
\end{aligned}$$

Rearranging the left-hand side of the above equation we obtain a bilateral generating function involving the Laguerre matrix polynomials and a certain terminating ${}_2F_1$ in the following theorem.

Theorem 2.5. *Let A and B be commuting matrices in $\mathbb{C}^{r\times r}$ satisfying the spectral conditions (7) and (8) and let λ be a complex number with $\Re(\lambda) > 0$, then the Laguerre matrix polynomials have the bilateral generating function*

$$\begin{aligned}
\sum_{n=0}^{\infty} L_n^{(A,\lambda)}(x)\ {}_2F_1(-nI, B; A+I; y)\, t^n &= (1-t)^{-(I+A-B)}(1-t+yt)^{-B}\\
&\times\exp\left(\frac{-\lambda xt}{1-t}\right)\ {}_1F_1\left(B; A+I; \frac{\lambda xyt}{(1-t)(1-t+yt)}\right).
\end{aligned}$$

It is easy to show that

$$\int_0^{\infty} e^{-tI}\left(\sqrt{xt}\right)^{A+nI} J_{A+nI}\left(2\sqrt{xt}\right)dt = e^{-xI}x^{A+nI}. \tag{23}$$

where $J_A(z)$ is the Bessel matrix functions defined by Jódar et al. (1994). Differentiating m times both sides of (23) with respect x and using the fact that

$$\frac{d}{dz}(z^A J_A(z)) = z^A J_{A-I}(z),$$

we get

$$\frac{d^m}{dx^m}\left(x^{A+nI}e^{-x}\right) = \int_0^\infty \left(\sqrt{xt}\right)^{A+nI-mI} J_{A+nI-mI}\left(2\sqrt{xt}\right)e^{-t}t^m dt,$$

for $m = 0, 1, 2, \ldots$.. Here, it is easy to justify the differentiation behind the integral sign. Setting $m = n$ in (10), we obtain the desired integral representation of the Laguerre matrix polynomials. These results are summarized below.

Theorem 2.6. *Let A be a matrix in $\mathbb{C}^{r\times r}$ satisfying the spectral condition (8) and let λ be a complex number with $\Re(\lambda) > 0$, then the Laguerre matrix polynomials satisfy the following integral representation:*

$$L_n^{(A,\lambda)}(x) = \frac{e^{xI}x^{\frac{-1}{2}A}}{n!}\int_0^\infty e^{-tI}t^{\frac{1}{2}A+nI}J_A\left(2\sqrt{xt}\right)dt.$$

3. Expansions of Laguerre matrix polynomials in a series of polynomials

In this section, we give expansion of the Laguerre matrix polynomials as series of Hermite and Legendre matrix polynomials.

Theorem 3.1. *Let A be a matrix in $\mathbb{C}^{r\times r}$ satisfying the spectral condition (8) and let λ be a complex number with $\Re(\lambda) > 0$, then the Laguerre matrix polynomials satisfy the following equations for $\|A\| > \frac{2}{\lambda}$:*

$$\begin{aligned} L_n^{(A,\lambda)}(x) = \sum_{k=0}^{n} & \frac{(-1)^k\lambda^k(\sqrt{2A})^{-k}}{k!\,(n-k)!}(A+I)_n\left[(A+I)_k\right]^{-1}H_k(x,A) \\ & \times {}_2F_2\left(\begin{array}{c} \frac{-(n-k)I}{2}, \frac{-(n-k-1)I}{2} \\ \frac{A+(k+1)I}{2}, \frac{A+(k+2)I}{2} \end{array}; \left(\frac{\lambda^2A^{-1}}{2}\right)\right), \end{aligned} \tag{24}$$

where $H_n(x,A)$ is the Hermite matrix polynomials, and

$$\begin{aligned} L_n^{(A,\lambda)}(x) = \sum_{k=0}^{n} & \frac{(-1)^k\lambda^k(2k+1)(\sqrt{2A})^{-k}}{(n-k)!2^k\left(\frac{3}{2}\right)_k}(A+I)_n\left[(A+I)_k\right]^{-1}P_k(x,A) \\ & \times {}_2F_3\left(\begin{array}{c} \frac{-(n-k)I}{2}, \frac{-(n-k-1)I}{2} \\ \frac{(3+2k)I}{2}, \frac{A+(k+1)I}{2}, \frac{A+(k+2)I}{2} \end{array}; \left(\frac{\lambda^2A^{-1}}{2}\right)\right), \end{aligned} \tag{25}$$

where $P_n(x,A)$ is the Legendre matrix polynomials.

Proof. By using the relations (16) and (9), we have

$$\sum_{n=0}^{\infty} L_n^{(A,\lambda)}(x)\, t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k (A+I)_{n+k} \left[(A+I)_k\right]^{-1} (\lambda x)^k}{k!n!} t^{n+k}.$$

On using the results given by Jódar and Company (1996)

$$(x\sqrt{2A})^k = \sum_{r=0}^{[\frac{1}{2}k]} \frac{k!}{r!(k-2r)!} H_{k-2r}(x,A)$$

and (15) we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} L_n^{(A,\lambda)}(x) t^n \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{[\frac{1}{2}k]} \frac{(-1)^k k!(A+I)_{n+k}[(A+I)_k]^{-1}\lambda^k(\sqrt{2A})^{-k}}{k!n!r!(k-2r)!} H_{k-2r}(x,A)t^{n+k} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{[\frac{n}{2}]} \frac{(-1)^k (A+I)_{n+k}[(A+I)_{k+2r}]^{-1}\lambda^{k+2r}(\sqrt{2A})^{-k-2r}}{(n-2r)!r!k!} H_k(x,A)t^{n+k}.
\end{aligned} \tag{26}$$

□

Substituting the well-known identity

$$\begin{aligned}
\frac{1}{(n-2r)!} I &= \frac{(-n)_{2r}}{n!} I \\
&= \frac{2^{2r}}{n!}(-\frac{nI}{2})_r(-\frac{(n-1)I}{2})_r; \quad \left(0 \le r \le \frac{1}{2}n\right)
\end{aligned}$$

and equation (3) in (26), we get proof of equation (24) after comparing the coefficients of t^n.

If we consider the equation

$$\frac{(x\sqrt{2A})^k}{k!} = \sum_{r=0}^{[\frac{1}{2}k]} \frac{(2k-4r+1)}{r!(\frac{3}{2})_{k-r}} P_{k-2r}(x,A),$$

given in (see Upadhyaya and Shehata (2011)), we get the proof of equation (25) similarly.

4. Generating matrix functions for Laguerre matrix polynomials

In this section, the interesting and alternative proofs of generating matrix functions for Laguerre matrix polynomials are derived. We prove here the following interesting formulae:

Theorem 4.1. *Let A and C be matrices in $\mathbb{C}^{r\times r}$ such that $C+(m+n)I$ is an invertible matrix for all integers $m+n\geq 0$ and $\Re(z)>-1$ for every eigenvalue $z\in\sigma(A)$ with $|z|<\frac{1}{4}$. Then the interesting generating matrix function for Laguerre matrix polynomials is*

$$H_6(-A,C;z,\lambda\, zw)=\sum_{n=0}^{\infty}(-A)_{2n}[(C)_n]^{-1}[(A+(1-2n)I)_n]^{-1}z^n L_n^{(A-2nI,\lambda)}(w), \tag{27}$$

where $A-2nI$ is a matrix in $\mathbb{C}^{r\times r}$ satisfying the spectral condition (8) and let λ be a complex number with $\Re(\lambda)>0$.

Proof. The L.H.S of (27) is equal to

$$\begin{aligned}H_6(-A,C;z,\lambda\, zw)&=\sum_{m,n=0}^{\infty}\frac{(-A)_{2n+m}[(C)_{n+m}]^{-1}}{n!m!}z^{n+m}(\lambda\, w)^m\\&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-A)_{2n-m}[(C)_n]^{-1}}{(n-m)!m!}z^n(\lambda\, w)^m\end{aligned}$$

by using (17), (9) and (4)

$$H_6(-A,C;z,zw)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-1)^m(-A)_{2n}[(I+A-2nI)_m]^{-1}[(C)_n]^{-1}}{m!(n-m)!}z^n(\lambda\, w)^m$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n}(-A)_{2n}[(C)_n]^{-1}[(I+A-2nI)_n]^{-1}z^n L_n^{(A-2nI,\lambda)}(w)$$

which proves (27). □

Theorem 4.2. *Let B and C be matrices in $\mathbb{C}^{r\times r}$ such that $C+(m+n)I$ is an invertible matrix for all integers $m+n\geq 0$ and $\Re(z)>-1$ for every eigenvalue $z\in\sigma(B)$. Then the Laguerre matrix polynomials have the following generating matrix function*

$$\Phi_3(-B;C;-z,-\lambda\, zw)=\sum_{m=0}^{\infty}[(C)_m]^{-1}z^m L_m^{(B-mI,\lambda)}(w), \tag{28}$$

where $B - mI$ is a matrix in $\mathbb{C}^{r\times r}$ satisfying the spectral condition (8) and let λ be a complex number with $\Re(\lambda) > 0$.

Proof. The L.H.S of (28) is equal to

$$\begin{aligned}\Phi_3(-B;C;-z,-\lambda\, zw) &= \sum_{m,n=0}^{\infty} \frac{(-1)^{m+n}(-B)_m[(C)_{m+n}]^{-1}}{m!n!} z^{m+n}(\lambda\, w)^n \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{m} \frac{(-1)^m(-B)_{m-n}[(C)_m]^{-1}}{(m-n)!n!} z^m(\lambda\, w)^n\end{aligned}$$

by using (17) and (9)

$$\begin{aligned}\Phi_3(-B;C;-z,-\lambda\, zw) &= \sum_{m,n=0}^{\infty} \frac{(-1)^{m+n}(-B)_m[(C)_{m+n}]^{-1}}{m!n!} z^{m+n}(\lambda\, w)^n \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{m} \frac{(-1)^m(-B)_{m-n}[(C)_m]^{-1}}{(m-n)!n!} z^m(\lambda\, w)^n \\ &= \sum_{m=0}^{\infty}[(C)_m]^{-1} z^m L_m^{(B-mI,\lambda)}(w)\end{aligned}$$

which proves (28). □

Theorem 4.3. *Let A_1, B_1 and C_2 be matrices in $\mathbb{C}^{r\times r}$ such that $C_2+(n+p)I$ is an invertible matrix for all integers $n+p \geq 0$ and $\Re(z) > -1$ for every eigenvalue $z \in \sigma(B_1)$ with $|z| < 1$. Laguerre matrix polynomials satisfy the following generating matrix function:*

$$\begin{aligned}&(1-z)^{-B_1}\ {}_3\Phi_M^{(4)}(A_1,-B_1,-B_1;A_1,C_2,C_2;z,-\lambda\, wu,-u(1-z)) \\ &= \sum_{p=0}^{\infty}[(C_2)_p]^{-1}u^p L_p^{(B_1-pI,\lambda)}(w),\end{aligned} \tag{29}$$

where $B_1 - pI$ is a matrix in $\mathbb{C}^{r\times r}$ satisfying the spectral condition (8) and let λ be a complex number with $\Re(\lambda) > 0$.

Proof. From (14), (17) and (9), it follows that

$$\begin{aligned}
&(1-z)^{-B_1}\ {}_3\Phi_M^{(4)}(A_1,-B_1,-B_1;A_1,C_2,C_2;z,-\lambda\,wu,-u(1-z))\\
&=\sum_{m,n,p=0}^{\infty}\frac{(-1)^{n+p}(-B_1)_{m+p}[(C_2)_{n+p}]^{-1}}{m!n!p!}z^m(1-z)^{pI-B_1}(\lambda\,w)^n u^{n+p}\\
&=\sum_{n,p=0}^{\infty}\frac{(-1)^{n+p}[(C_2)_{n+p}]^{-1}}{n!p!}(1-z)^{pI-B_1}(\lambda\,w)^n u^{n+p}\sum_{m=0}^{\infty}\frac{(-B_1)_{m+p}}{m!}z^m\\
&=\sum_{n,p=0}^{\infty}\frac{(-1)^{n+p}[(C_2)_{n+p}]^{-1}}{n!p!}(1-z)^{pI-B_1}(\lambda\,w)^n u^{n+p}\sum_{m=0}^{\infty}\frac{(-B_1)_p(-B_1+pI)_m}{m!}z^m\\
&=\sum_{n,p=0}^{\infty}\frac{(-1)^{n+p}(-B_1)_p[(C_2)_{n+p}]^{-1}}{n!p!}(1-z)^{pI-B_1}(\lambda\,w)^n u^{n+p}(1-z)^{B_1-pI}\\
&=\sum_{n,p=0}^{\infty}\frac{(-1)^{n+p}(-B_1)_p[(C_2)_{n+p}]^{-1}}{n!p!}(\lambda\,w)^n u^{n+p}\\
&=\sum_{p=0}^{\infty}\sum_{n=0}^{p}\frac{(-1)^{p}(-B_1)_{p-n}[(C_2)_{p}]^{-1}}{n!(p-n)!}(\lambda\,w)^n u^{p}\\
&=\sum_{p=0}^{\infty}[(C_2)_p]^{-1}u^p L_p^{(B_1-pI,\lambda)}(w).
\end{aligned}$$

□

Theorem 4.4. *Let A_1, B_1, B_2 and C_2 be matrices in $\mathbb{C}^{r\times r}$ such that $C_2+(n+p)I$ is an invertible matrix for all integers $n+p\geq 0$ and $\Re(z)>-1$ for every eigenvalue $z\in\sigma(B_2)$ with $|z|<1$ and $|w(z-1)|<1$. Laguerre matrix polynomials satisfy the following relation:*

$$\begin{aligned}
&(1-z)^{A_1}\ {}_3\Phi_G^{(1)}(A_1,A_1,A_1,B_1,-B_2;B_1,C_2,C_2;z,w(z-1),\lambda\,wu(z-1))\\
&=\sum_{n=0}^{\infty}(A_1)_n[(C_2)_n]^{-1}w^n L_n^{(B_2-nI,\lambda)}(u),
\end{aligned} \tag{30}$$

where B_2-nI is a matrix in $\mathbb{C}^{r\times r}$ satisfying the spectral condition (8) and let λ be a complex number with $\Re(\lambda)>0$.

Proof. Using (14), (17) and (9), we have

$$
\begin{aligned}
&(1-z)^{A_1}\ {}_3\Phi_G^{(1)}(A_1,A_1,A_1,B_1,-B_2;B_1,C_2,C_2;z,w(z-1),\lambda\, wu(z-1))\\
&=\sum_{m,n,p=0}^{\infty}\frac{(-1)^{n+p}(A_1)_{m+n+p}(-B_2)_n[(C_2)_{n+p}]^{-1}}{m!n!p!}z^m(1-z)^{A_1+(n+p)I}w^{n+p}(\lambda\, u)^p\\
&=\sum_{n,p=0}^{\infty}\frac{(-1)^{n+p}(-B_2)_n[(C_2)_{n+p}]^{-1}}{n!p!}(1-z)^{A_1+(n+p)I}w^{n+p}(\lambda\, u)^p\sum_{m=0}^{\infty}\frac{(A_1)_{m+n+p}}{m!}z^m\\
&=\sum_{n,p=0}^{\infty}\frac{(-1)^{n+p}(A_1)_{n+p}(-B_2)_n[(C_2)_{n+p}]^{-1}}{n!p!}w^{n+p}(\lambda\, u)^p\\
&=\sum_{n=0}^{\infty}\sum_{p=0}^{n}\frac{(-1)^{n}(A_1)_{n}(-B_2)_{n-p}[(C_2)_{n}]^{-1}}{(n-p)!p!}w^{n}(\lambda\, u)^p\\
&=\sum_{n=0}^{\infty}(A_1)_n[(C_2)_n]^{-1}w^nL_n^{(B_2-nI,\lambda)}(u).
\end{aligned}
$$

□

Theorem 4.5. *Let B be a matrix in $\mathbb{C}^{r\times r}$, $B-mI$ satisfying the spectral condition (8) and let λ be a complex number with $\Re(\lambda)>0$. Then Laguerre matrix polynomials can be written as hypergeometric matrix function:*

$$
\begin{aligned}
&\sum_{m=0}^{\infty}\frac{1}{m!}\ {}_0F_1(-;mI-B;\lambda\, zw)z^m\\
&=\sum_{m=0}^{\infty}\Gamma(B+(1-m)I)\Gamma^{-1}(B+I)z^mL_m^{(B-mI,\lambda)}(w).
\end{aligned}
\tag{31}
$$

Proof. Putting $B=C$ in (13), we obtain

$$
\begin{aligned}
\Phi_3(-B;-B;-z,-\lambda\, zw)&=\sum_{m,n=0}^{\infty}\frac{(-1)^{m+n}(-B)_m[(-B)_{m+n}]^{-1}}{m!n!}z^{m+n}(\lambda\, w)^n\\
&=\sum_{m,n=0}^{\infty}\frac{(-1)^{m+n}[(mI-B)_n]^{-1}}{m!n!}z^{m+n}(\lambda\, w)^n\\
&=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}z^m\sum_{n=0}^{\infty}\frac{(-1)^n[(mI-B)_n]^{-1}}{n!}z^n(\lambda\, w)^n\\
&=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}z^m\ {}_0F_1(-;mI-B;-\lambda\, zw)
\end{aligned}
$$

by using (28), we obtain (31). □

Theorem 4.6. *Let B_2 is a matrix in $\mathbb{C}^{r\times r}$, $B_2 - nI$ satisfy the condition (8) and let λ be a complex number with $\Re(\lambda) > 0$. Laguerre matrix polynomials satisfy the interesting generating matrix function:*

$$e^{-\lambda\, zw}(1+w)^{B_2} = \sum_{n=0}^{\infty} w^n L_n^{(B_2-nI,\lambda)}(z). \tag{32}$$

Proof. The L.H.S. of (32) is equal to

$$\begin{aligned}
e^{-\lambda\, zw}(1+w)^{B_2} &= \sum_{n,p=0}^{\infty} \frac{(-1)^{n+p}(-B_2)_n}{n!p!} w^{n+p}(\lambda\, z)^p \\
&= \sum_{n,p=0}^{\infty} \frac{(-1)^{n+p}(-B_2)_n}{n!p!} w^{n+p}(\lambda\, z)^p = \sum_{n=0}^{\infty}\sum_{p=0}^{n} \frac{(-1)^{n}(-B_2)_{n-p}}{(n-p)!p!} w^{n}(\lambda\, z)^p \\
&= \sum_{n=0}^{\infty} w^n L_n^{(B_2-nI,\lambda)}(z)
\end{aligned}$$

which proves (32). □

Let A be a matrix in $\mathbb{C}^{N\times N}$ satisfying the spectral condition (8) and let λ be a complex number with $\Re(\lambda) > 0$. We define the generalized Laguerre matrix polynomials of two variables by the following

$$L_n^{(A,\lambda)}(x,y) = \sum_{k=0}^{n} \frac{(-1)^k (A+I)_n [(A+I)_k]^{-1} \lambda^k y^{n-k} x^k}{k!(n-k)!}. \tag{33}$$

Using (3), (16) and (33), we obtain the generating matrix function which represents an explicit representation for the Laguerre matrix polynomials

$$\sum_{n=0}^{\infty} L_n^{(A,\lambda)}(x,y) t^n = (1-yt)^{-(A+I)} \exp\left(\frac{-\lambda\, xt}{1-yt}\right). \tag{34}$$

Finally, it is now interesting to extend the above results to new generalized forms of generalized Laguerre matrix polynomials of two variables can be defined in the form:

$$L_n^{(A,\lambda)}(x,y;B) = \sum_{k=0}^{n} \frac{(-1)^k (A+I)_n [(A+I)_k]^{-1} \mathbb{B}_{n,k} \lambda^k y^{n-k}}{k!(n-k)!} \tag{35}$$

where

$$\mathbb{B}_{n,k} = n!x^k\ {}_2F_1(-kI, B; -nI; x) = \sum_{i=0}^{k} \frac{k!(n-i)!}{i!(k-i)!}\,(B)_i x^{k+i}, \tag{36}$$

where A and B are matrices in $\mathbb{C}^{N\times N}$ such that A satisfies the condition (8) and B satisfies the condition $\Re(z) > 0$ for every eigenvalue $z \in \sigma(B)$.

When B is the zero matrix, then the Laguerre matrix polynomials of two variables reduce to

$$L_n^{(A,\lambda)}(x, y; \mathbf{0}) = L_n^{(A,\lambda)}(x, y) \tag{37}$$

From (36), we can write in the following integral representation

$$\mathbb{B}_{n,k} = x^k\Gamma^{-1}(B)\int_0^\infty\int_0^\infty e^{-(t+u)}t^n u^{B-I}(1+\frac{xu}{t})^k dtdu. \tag{38}$$

Theorem 4.7. *Let A and B be matrices in $\mathbb{C}^{N\times N}$ such that A satisfy the condition (8) and B satisfy the condition $\Re(z) > 0$ for every eigenvalue $z \in \sigma(B)$. Then the generalized Laguerre matrix polynomials of two variables has the following integral representation:*

$$L_n^{(A,\lambda)}(x, y; B) = \Gamma^{-1}(B)\int_0^\infty\int_0^\infty e^{-(t+u)}t^n u^{B-I} L_n^{(A,\lambda)}(x(1+\frac{xu}{t}), y)dtdu. \tag{39}$$

Proof. Using (33), (38) and (35), we obtain (39). Thus, the result is completed. □

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